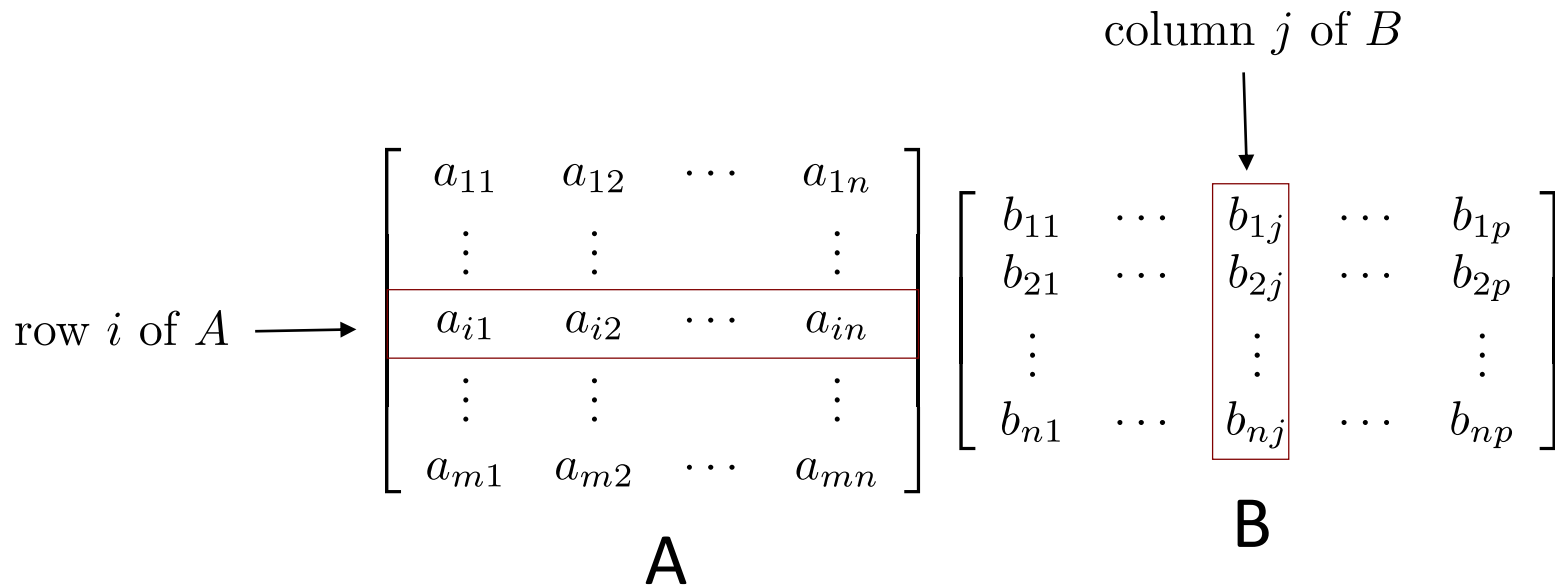


Matrix Multiplication

Hung-yi Lee

Matrix Multiplication

- Given two matrices A and B , the (i, j) -entry of AB is the inner product of **row i of A** and **column j of B**



$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Matrix Multiplication

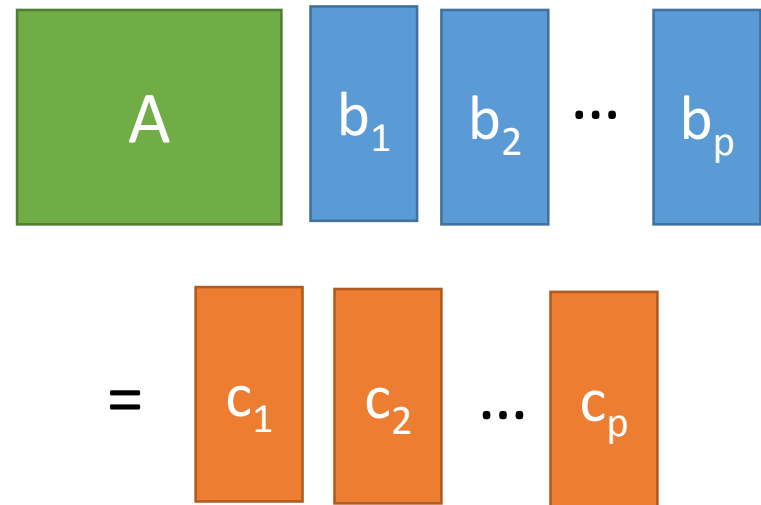
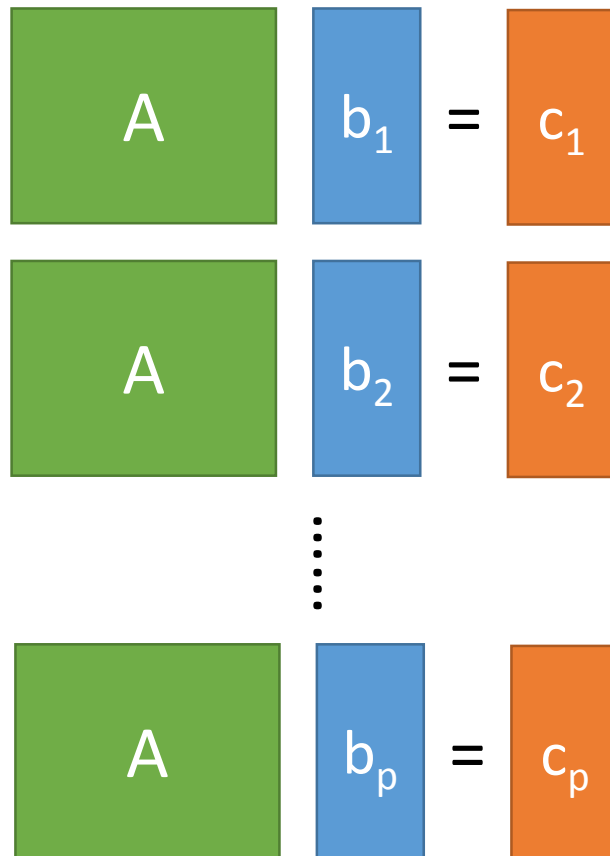
- Given two matrices A and B , the (i, j) -entry of AB is the inner product of **row i of A** and **column j of B**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{C = AB} = \begin{bmatrix} (-1) \times 1 + 3 \times 2 & 1 \times 1 + 2 \times 2 \\ (-1) \times 3 + 3 \times 4 & 1 \times 3 + 2 \times 4 \\ (-1) \times 5 + 3 \times 6 & 1 \times 5 + 2 \times 6 \end{bmatrix}$$

Matrix Multiplication - Meaning

- Multiple Input

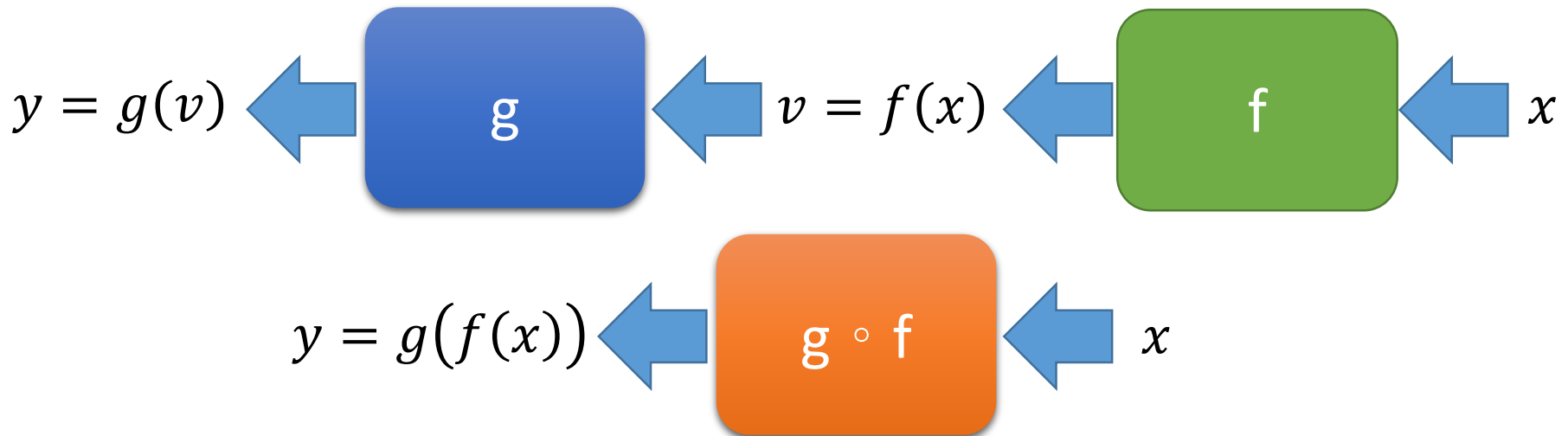


$$AB = A[b_1 \quad b_2 \quad \dots \quad b_p]$$
$$= [Ab_1 \quad Ab_2 \quad \dots \quad Ab_p]$$

Matrix Multiplication - Meaning

- **Composition**

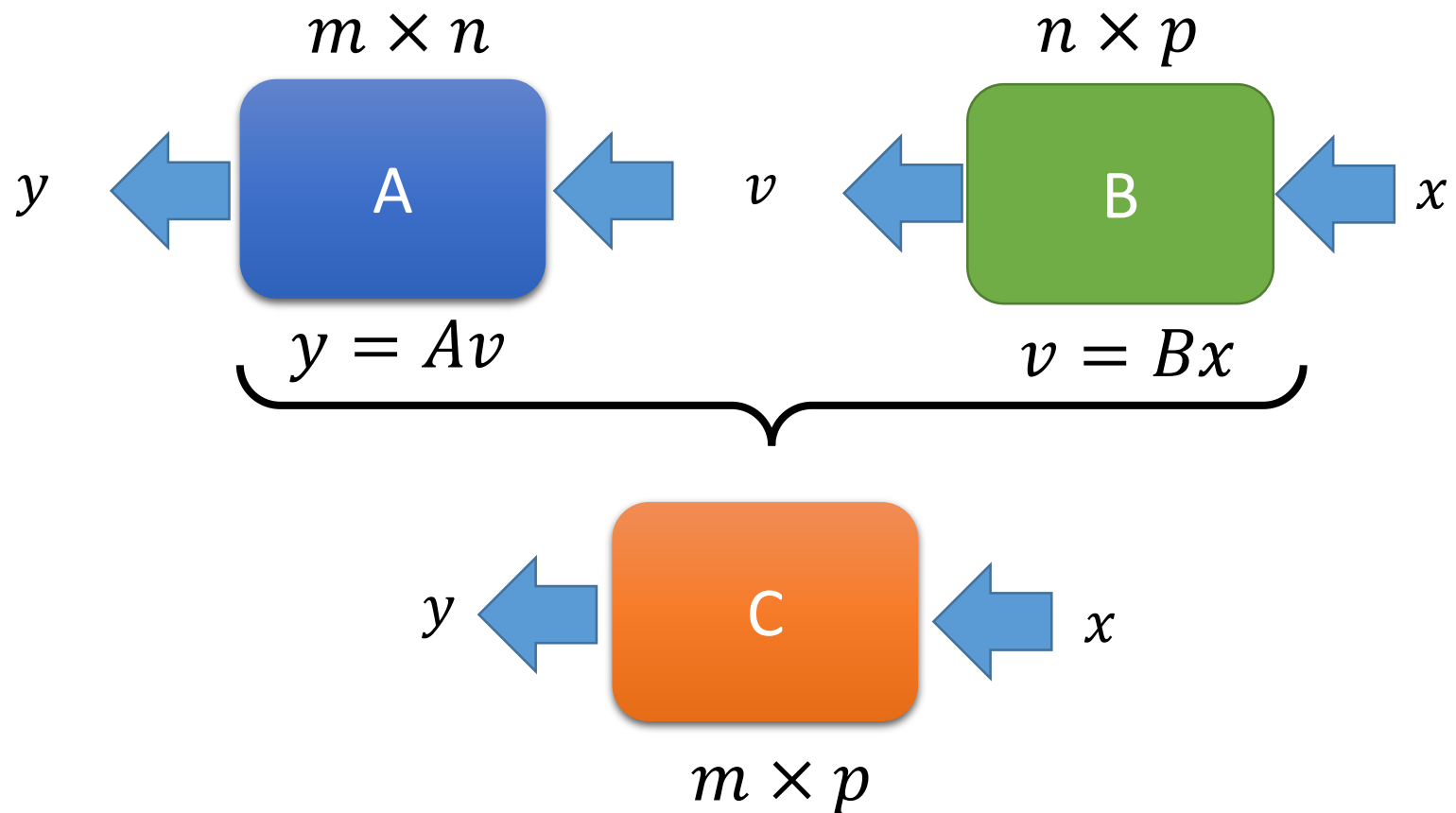
- Given two function f and g , the function $g(f(\cdot))$ is the composition $g \circ f$.



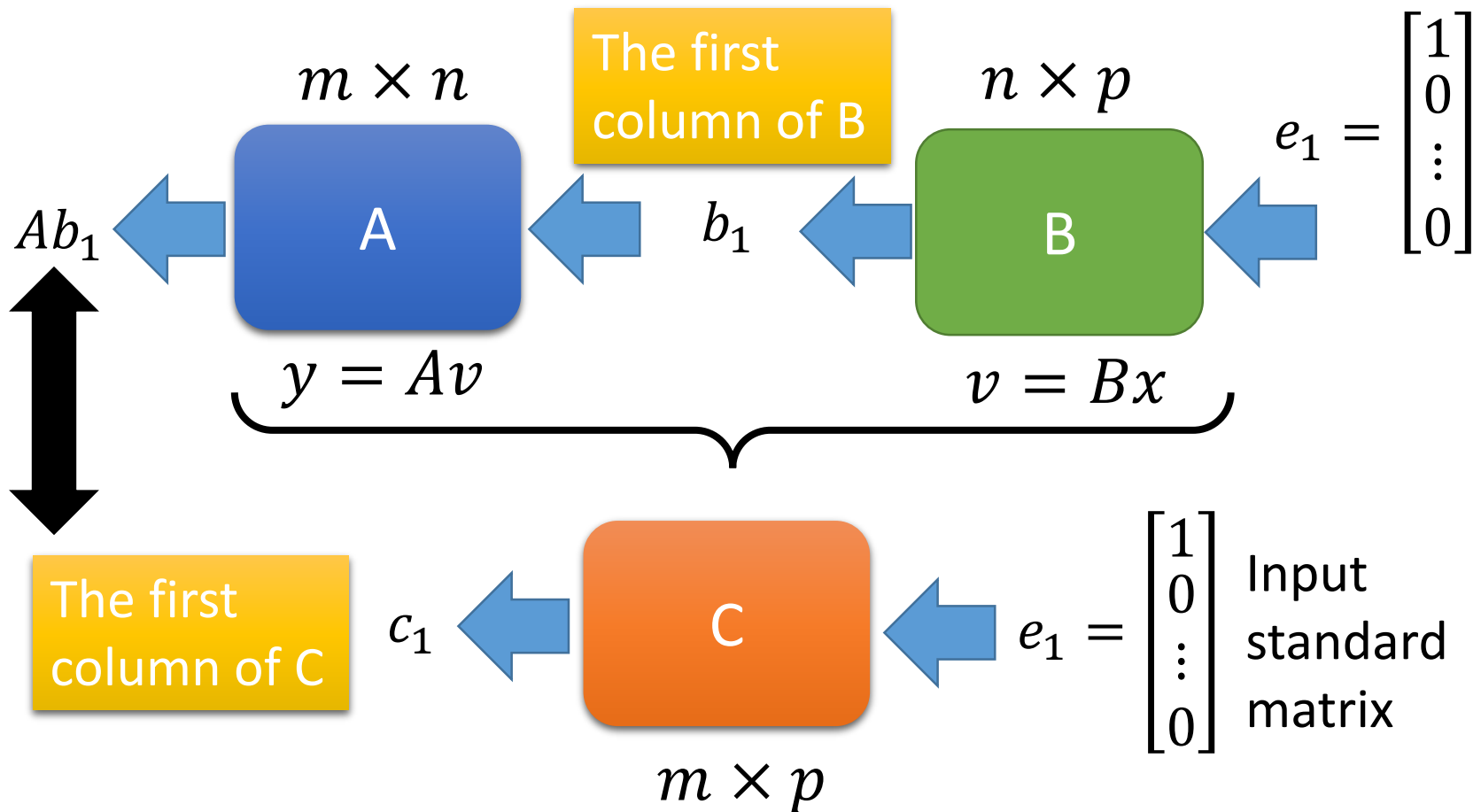
Matrix multiplication is the composition of two linear functions.

Matrix Multiplication - Meaning

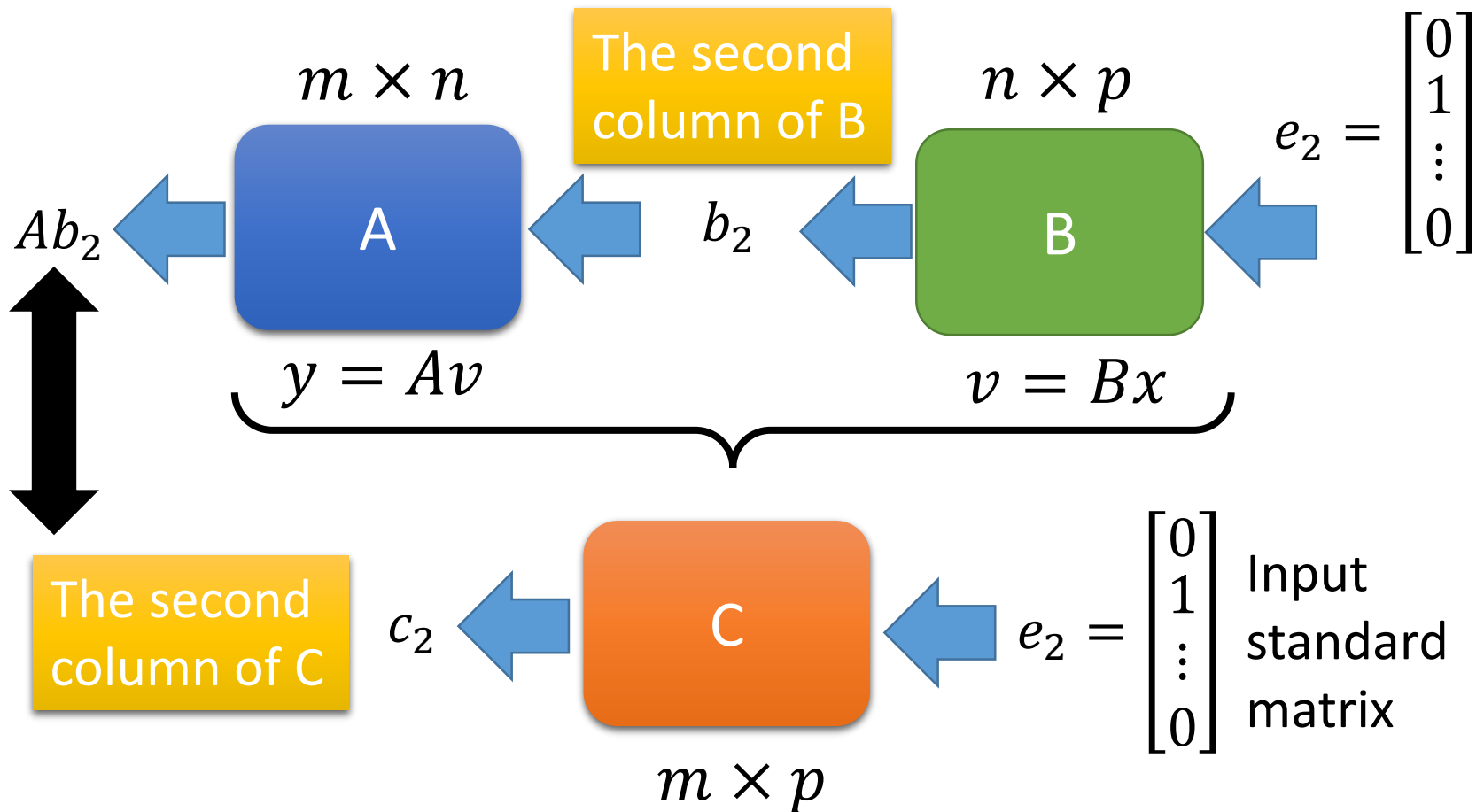
- Composition

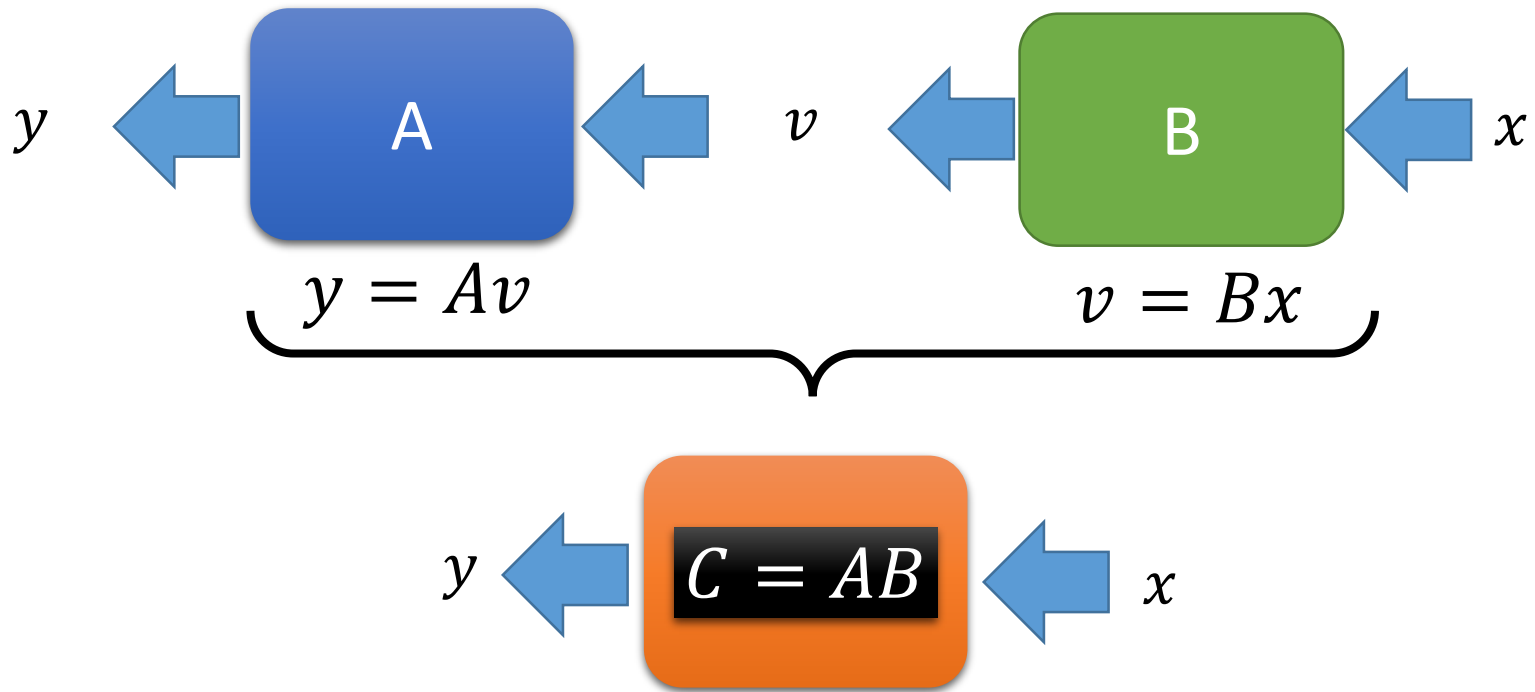


Matrix Multiplication - Meaning



Matrix Multiplication - Meaning





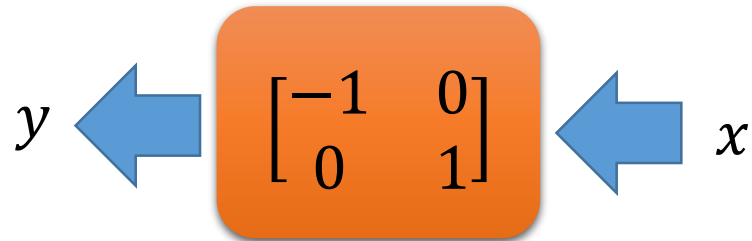
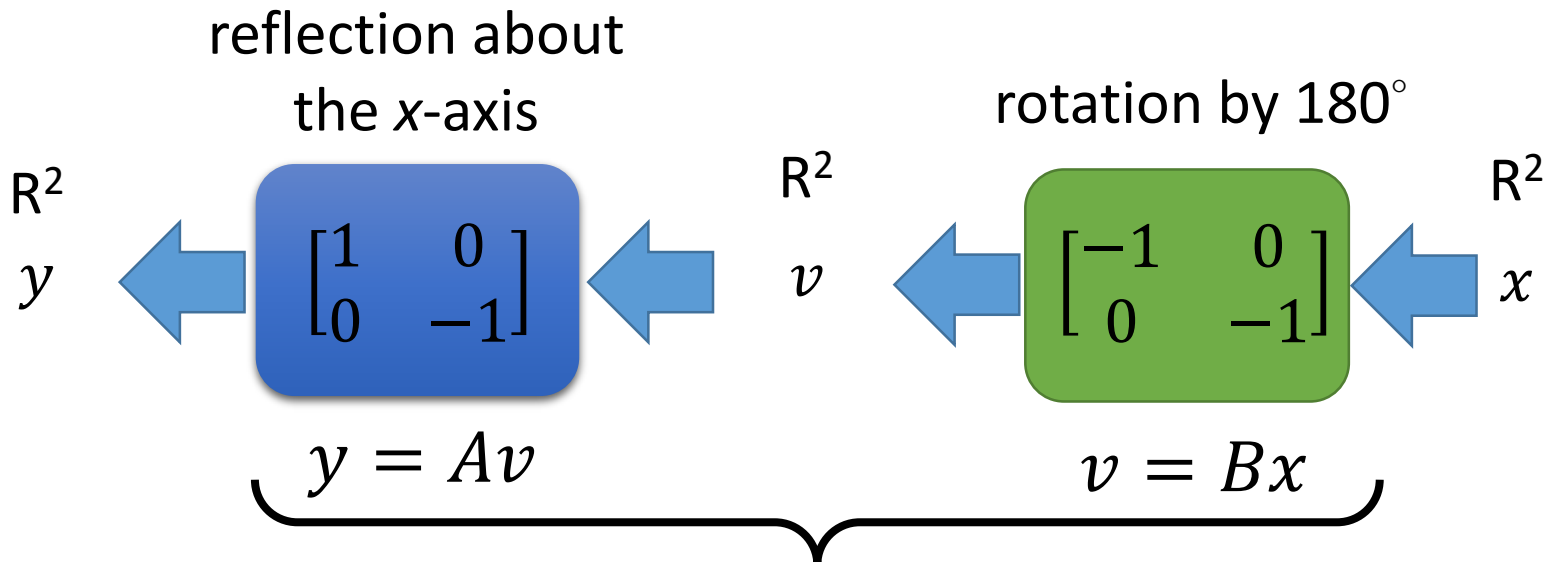
The composition of A and B is

$$C = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$



reflection about the y-axis

Not Commutative

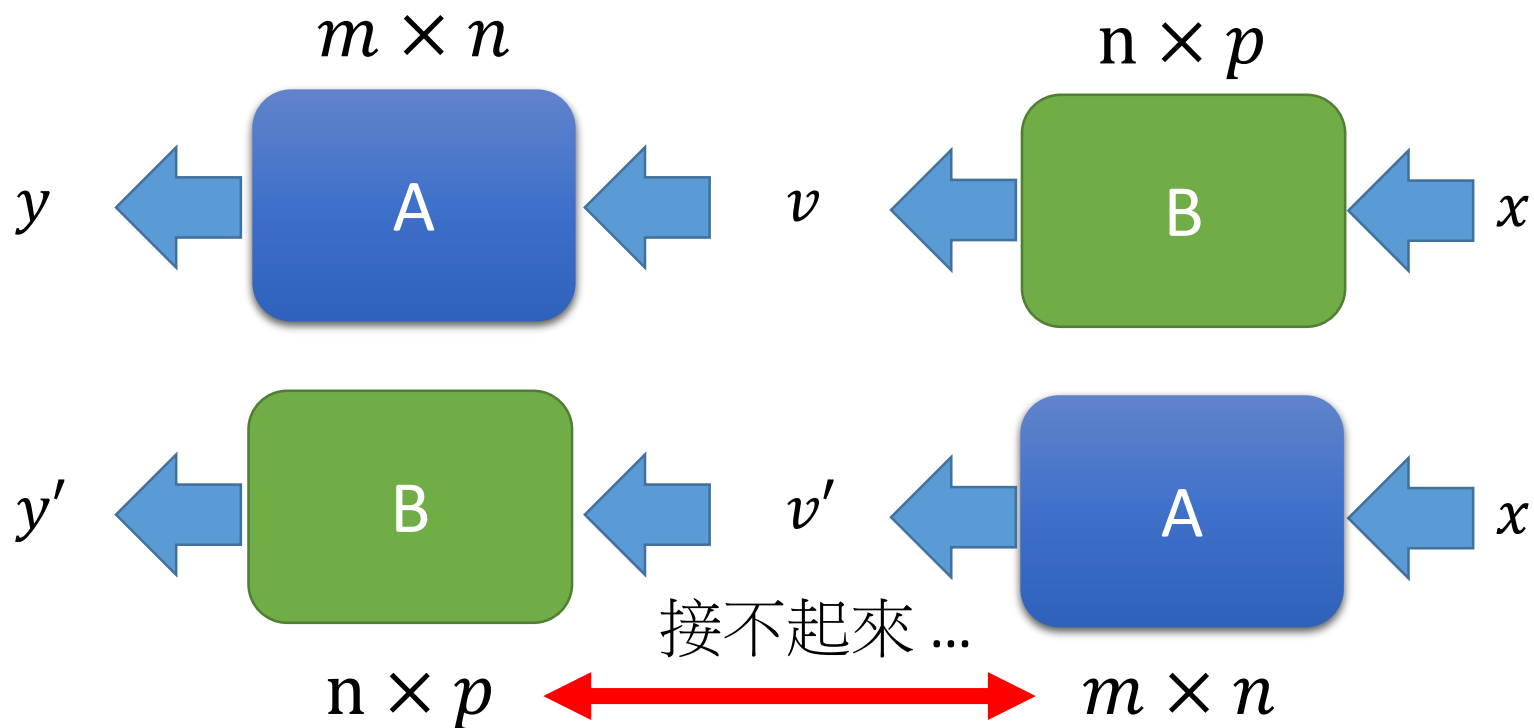
- $AB \neq BA$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\neq BA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Not Communicative



If A and B are matrices, then both AB and BA are defined if and only if A and B are square matrices?

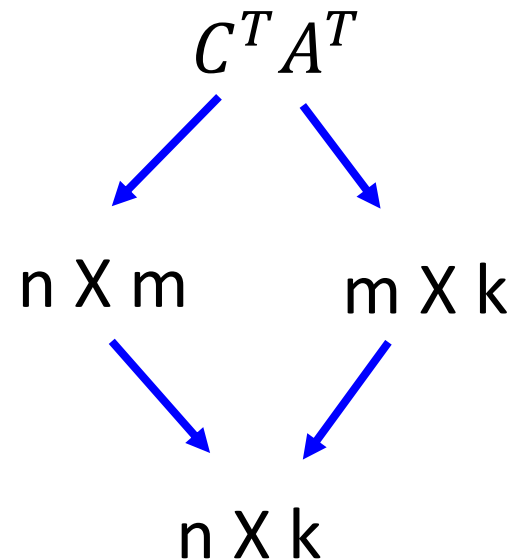
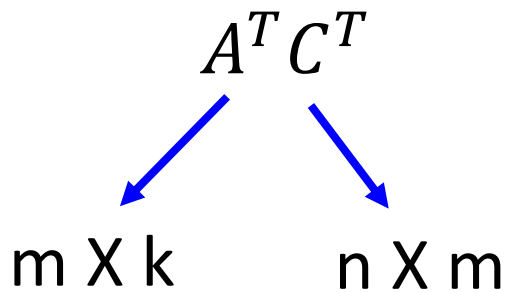
Properties

- Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q be $n \times p$ matrices
 - For any scalar s , $s(AC) = (sA)C = A(sC)$
 - $(A + B)C = AC + BC$
 - $C(P+Q)=CP+CQ$
 - $I_k A = A = A I_m$
 - The product of any matrix and a zero matrix is a zero matrix
- Power of square matrices: $A \in \mathcal{M}_{n \times n}$, $A^k = A A \cdots A$ (k times), and by convention, $A^1 = A$, $A^0 = I_n$.

Properties

$$AC: k \times n \quad (AC)^T: n \times k$$

- Let A be $k \times m$ matrices, C be an $m \times n$ matrix,
 - $(AC)^T = ? \quad C^T A^T$



Special Matrix

- Diagonal Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

- Symmetric Matrix $A^T = A$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix} = A^T \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq B^T$$

AA^T and $A^T A$ are square and symmetric

$$(AA^T)^T = A^{TT} A^T = AA^T \quad (A^T A)^T = A^T A^{TT} = A^T A$$

Augmentation and Partition

- Augment: the augment of A and B is [A B]
- Partition:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \left[\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right]$$

Multiply as the small matrices are scalar

Don't switch the order

Block Multiplication

$$A = \begin{bmatrix} \boxed{1} & \boxed{3} & \boxed{4} & \boxed{2} \\ \boxed{0} & \boxed{5} & \boxed{-1} & \boxed{6} \\ \boxed{1} & \boxed{0} & \boxed{3} & \boxed{-1} \end{bmatrix}$$

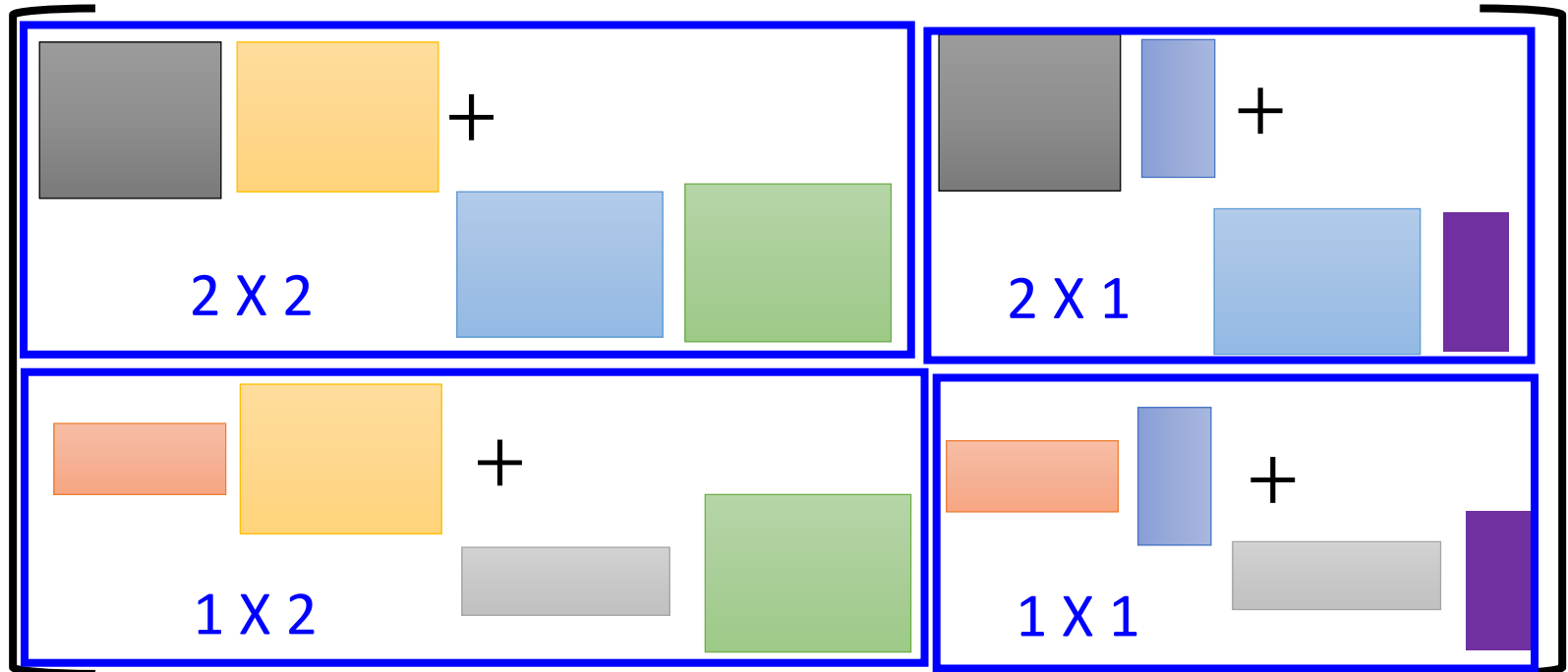
"2 x 2"

$$B = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{0} \\ \boxed{2} & \boxed{-1} & \boxed{2} \\ \boxed{0} & \boxed{3} & \boxed{1} \end{bmatrix}$$

"2 x 2"

$AB =$

"2 x 2"



Block Multiplication

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 6 & 8 & 5 & 0 \\ -7 & 9 & 0 & 5 \end{array} \right] \quad A = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 8 \\ -7 & 9 \end{bmatrix}$$

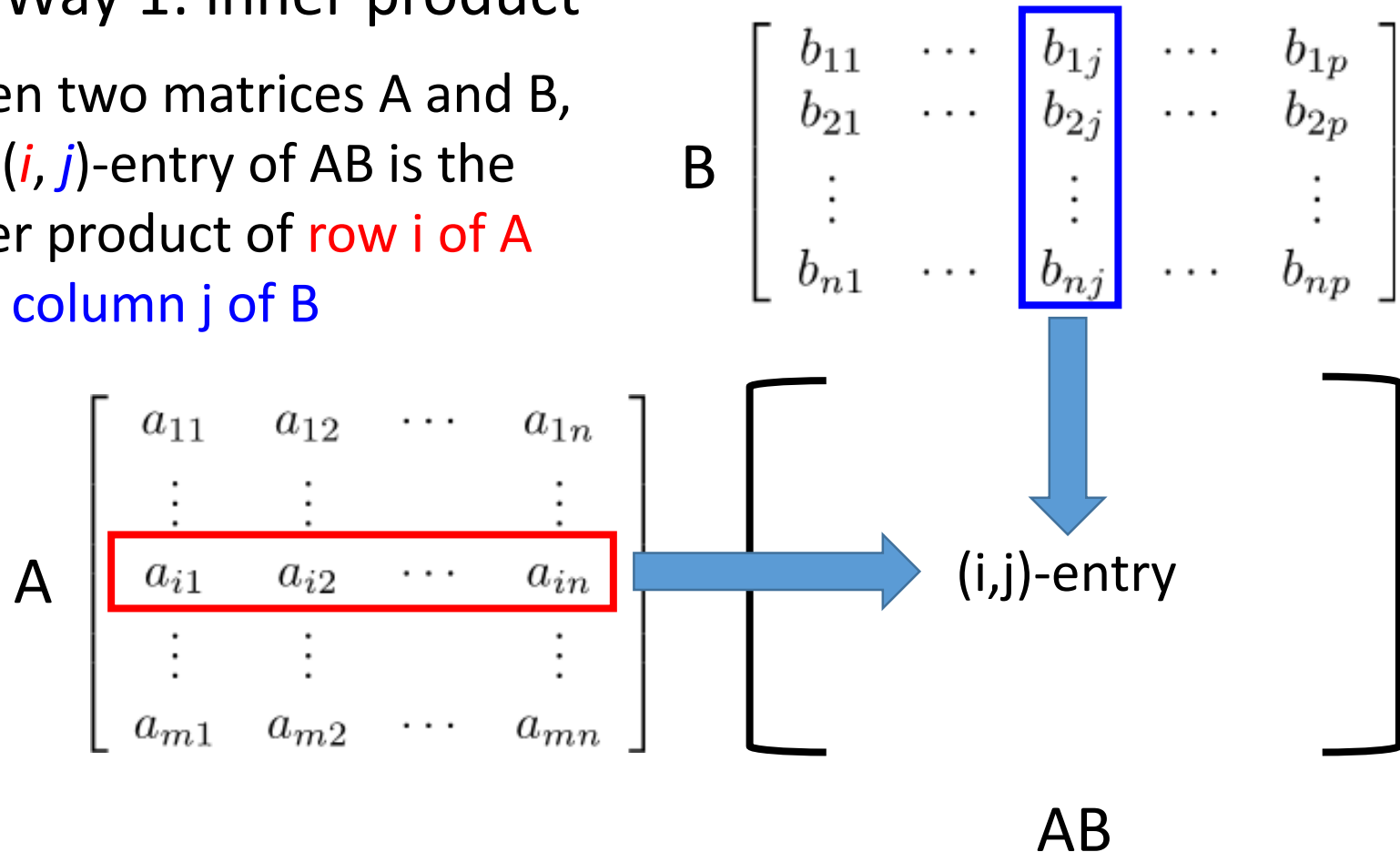
$$A^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \begin{bmatrix} I_2 & O \\ 6B & 25I_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Matrix Multiplication – 4 way

- Way 1: inner product

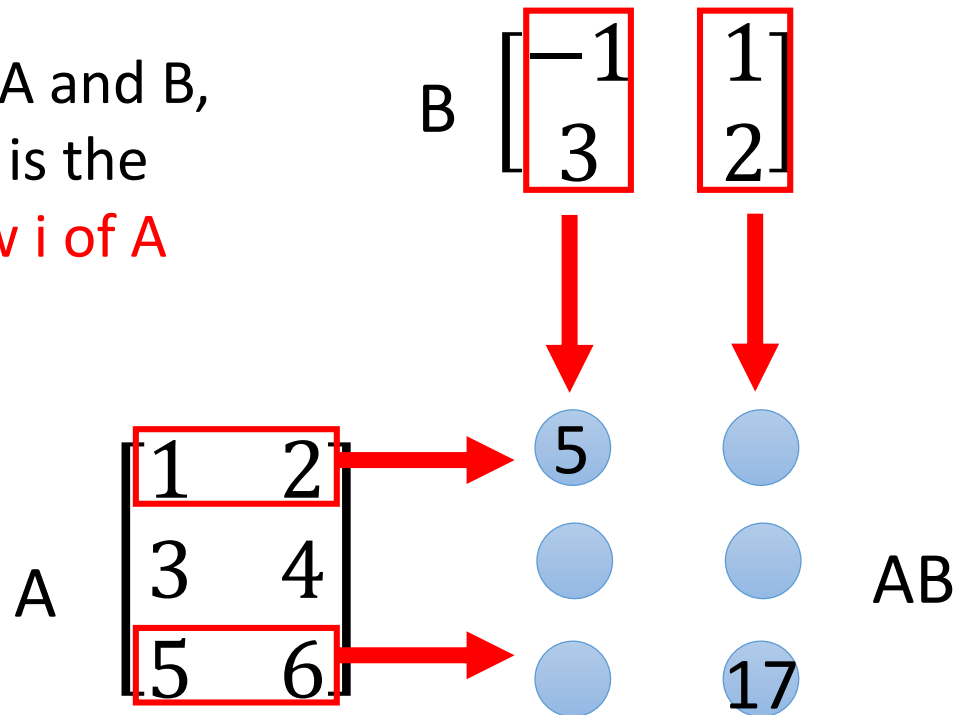
Given two matrices A and B, the (i, j) -entry of AB is the inner product of **row i of A** and **column j of B**



Matrix Multiplication – 4 way

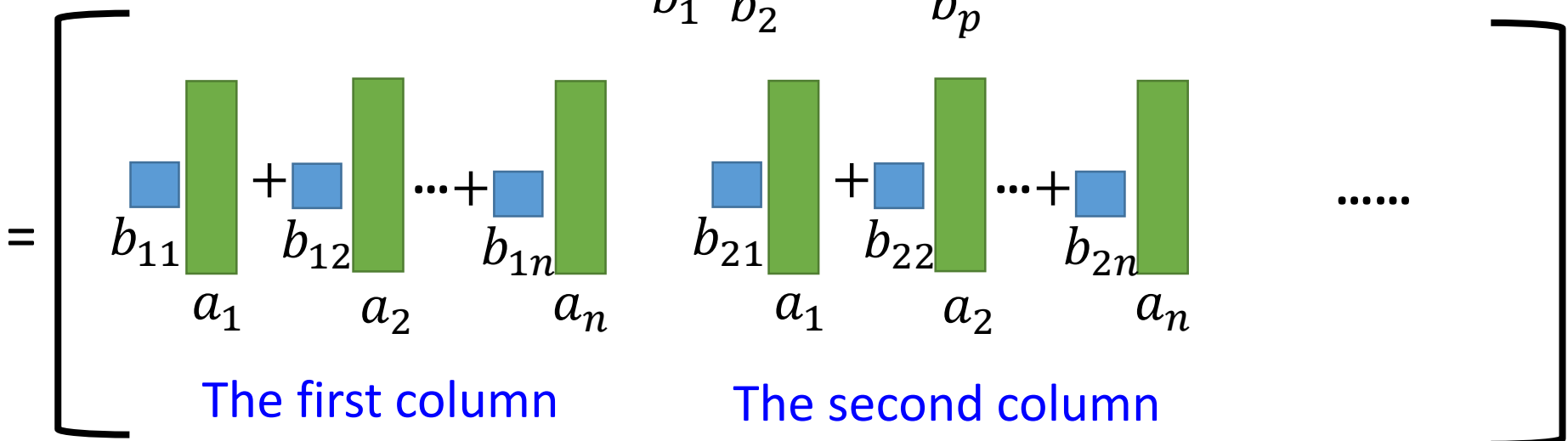
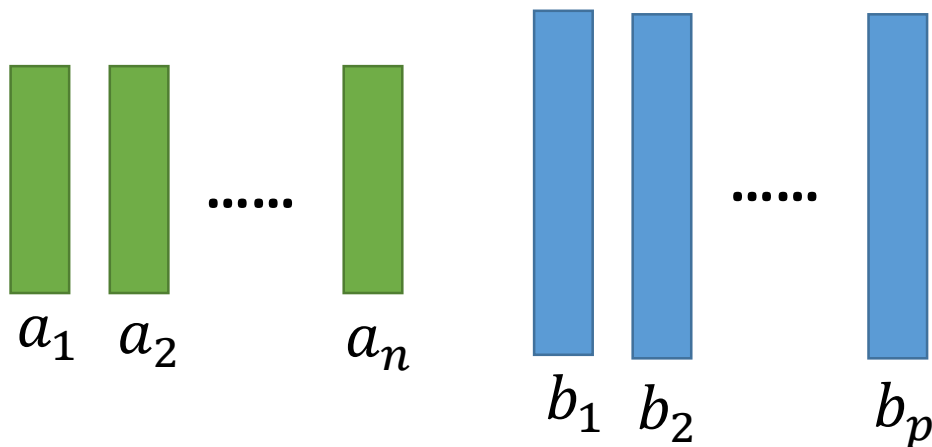
- Way 1: inner product

Given two matrices A and B,
the (i, j) -entry of AB is the
inner product of **row i of A**
and **column j of B**



Matrix Multiplication – 4 way

- Way 2: Linear combination of columns



Matrix Multiplication – 4 way

- Way 2: Linear combination of columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \left[\begin{array}{c} -1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \\ \text{The first column} \end{array} \quad \begin{array}{c} 1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \\ \text{The second} \\ \text{column} \end{array} \right]$$

Matrix Multiplication – 4 way

- View 3: Linear combination of rows

The diagram illustrates the process of matrix multiplication as a linear combination of rows. On the left, a matrix is represented by a column of row vectors $a_1^T, a_2^T, \dots, a_m^T$ (green bars) multiplied by a column of column vectors $b_1^T, b_2^T, \dots, b_n^T$ (blue bars). An equals sign follows, leading to a large bracketed column of row vectors representing the result. Each row in the result is a linear combination of the b_j^T vectors, with coefficients from the corresponding row of the a_i^T vectors.

$$\begin{bmatrix} a_1^T & a_2^T & \vdots & a_m^T \end{bmatrix} \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix} = \begin{bmatrix} a_{11}b_1^T + a_{12}b_2^T \cdots + a_{1n}b_n^T \\ a_{21}b_1^T + a_{22}b_2^T \cdots + a_{2n}b_n^T \\ \vdots \\ a_{m1}b_1^T + a_{m2}b_2^T \cdots + a_{mn}b_n^T \end{bmatrix}$$

Matrix Multiplication – 4 way

- Way 3: Linear combination of rows

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1[-1 \quad 1] + 2[3 \quad 2] \\ 3[-1 \quad 1] + 4[3 \quad 2] \\ 5[-1 \quad 1] + 6[3 \quad 2] \end{bmatrix}$$

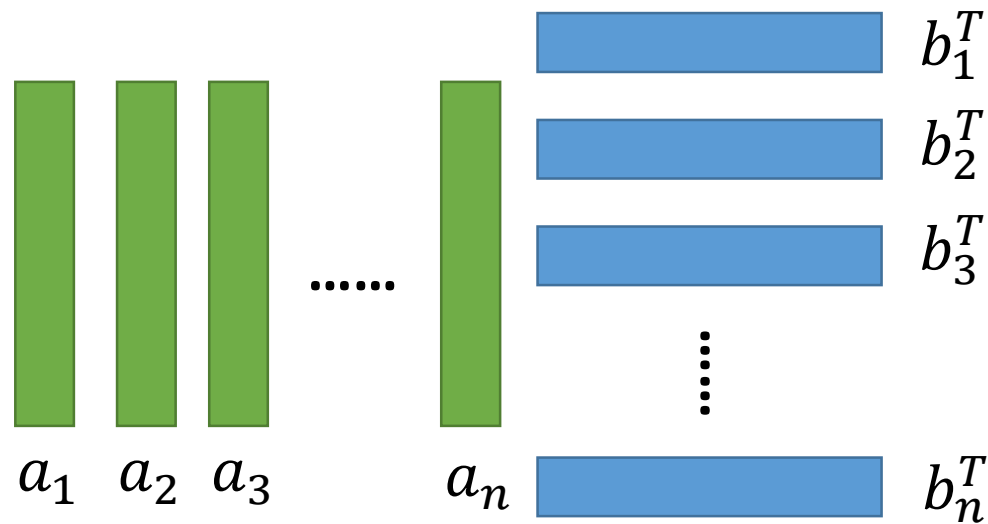
The first row

The second row

The third row

Matrix Multiplication: 4 views

- View 4: Summation of matrices



$$= a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

matrices

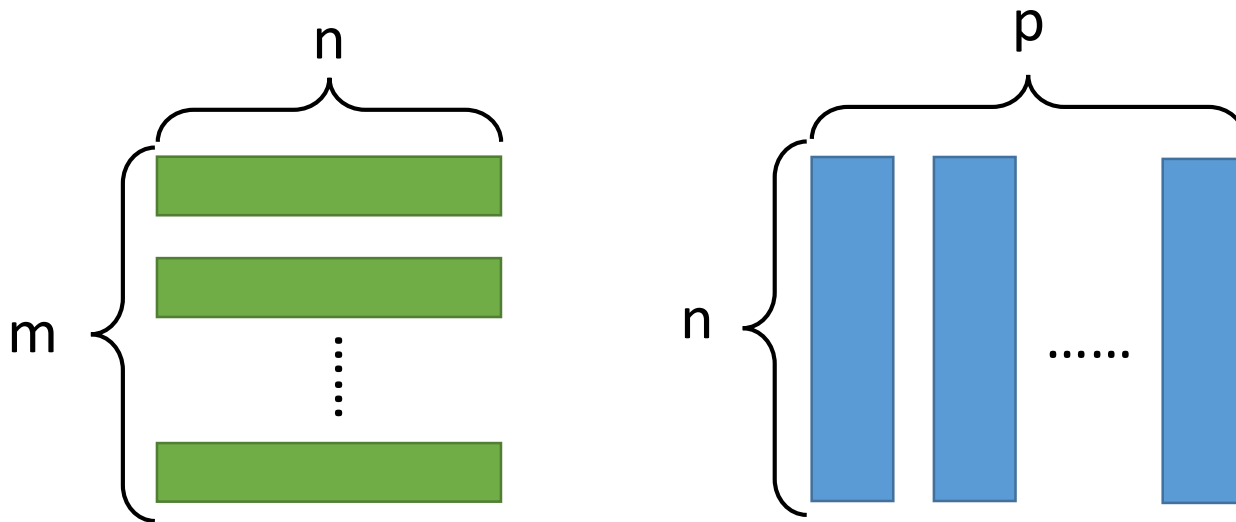
Matrix Multiplication: 4 views

- Way 4: summation of matrices

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} \\ \text{"1 x 2"} & \quad \text{"2 x 1"} \\ & \quad \quad \quad \text{"1 x 1"} \end{aligned} \\ &= \begin{bmatrix} \begin{bmatrix} -1 & 1 \\ -3 & 3 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 12 & 8 \\ 18 & 12 \end{bmatrix} \\ \text{Rank = ?} & \quad \quad \quad \text{Rank = ?} \end{bmatrix} \end{aligned}$$

Practical Issue

- Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q be $n \times p$ matrices
 - $A(CP) = (AC)P$



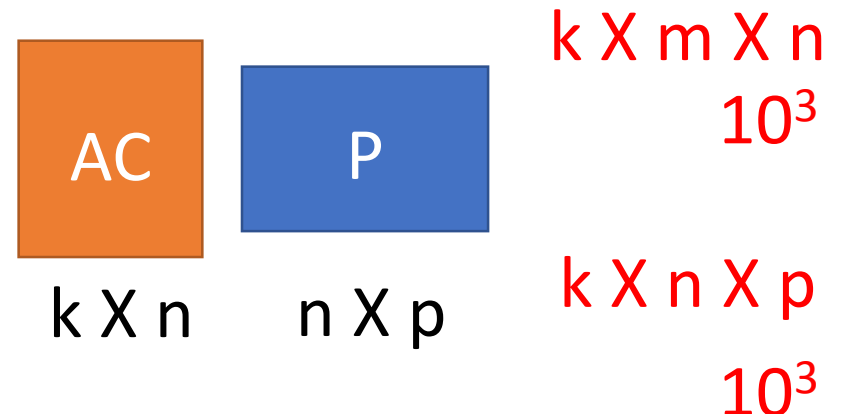
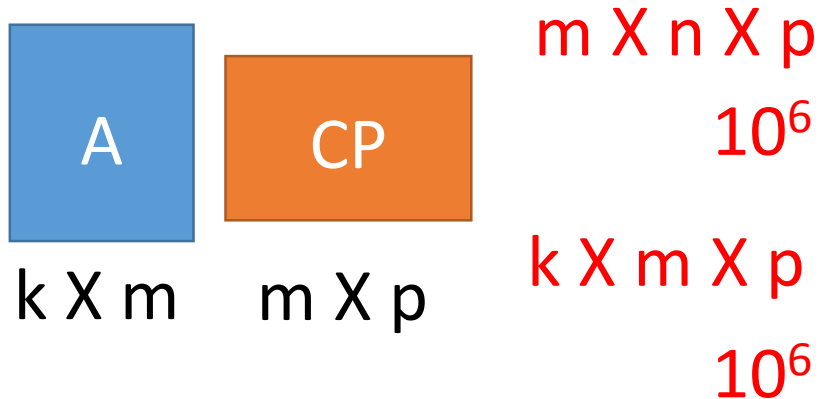
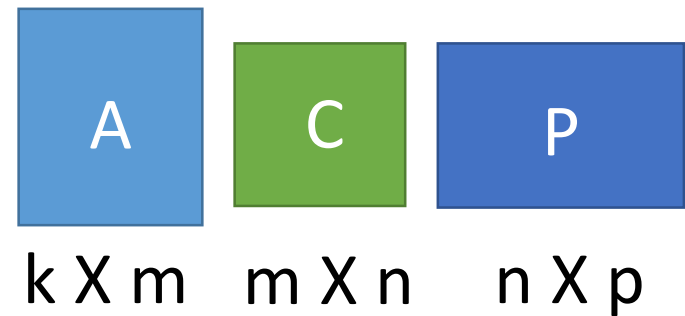
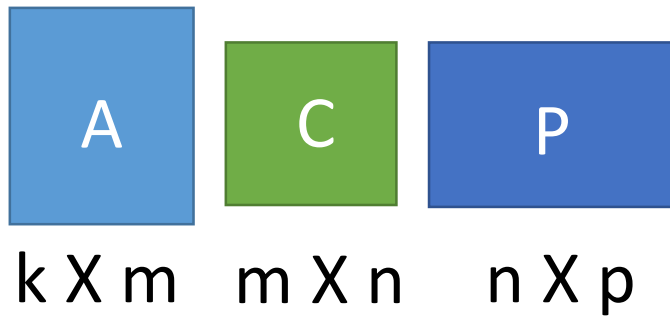
Multiplication count: $m \times n \times p$

Practical Issue

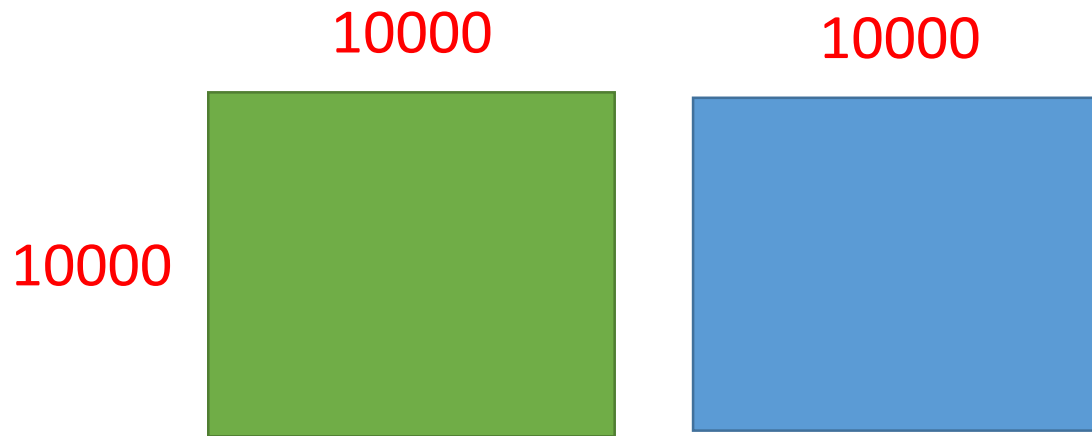
k=1 m=1000

n=1 p=1000

- Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q be $n \times p$ matrices
 - $A(CP) = (AC)P$



Practical Issue - GPU



Multiplying two 10000 X 10000 matrices

CPU `It cost 21.249996 sec`

GPU `It cost 0.843893 sec`

(GTX 980 Ti)

More than 20 times faster

Acknowledgement

- 感謝 陶昇永 同學發現投影片上的錯誤